4.13 Taylor's formula with remainder

Now we get to the end, the remnant, the stuff that's left over when everything else has been snapped up. Yes, it's time for the remainder sale. Time to move out those Taylor series remainders.

Although Taylor's formula gives us a good approximation to $f(x)$ near $x = a$, it's not exact, and there is an error. And sometimes when we use Taylor's formula we absolutely positively have to know how large the error can be. Otherwise, the bridge we designed almost holds cars. Or the new male contraceptive pill we have created almost prevents pregnancy. Fortunately, there is a simple method for bounding the error.

If we take a degree n Taylor polynomial to approximate $f(x)$ near $x = a$, we get:

Taylor's Formula with Remainder

$$f(x) = \underbrace{f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n}_{\text{Taylor polynomial } P_n(x)} + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}}_{\text{Remainder } R_n(x)}$$

where c is some number between a and x.

The quantity $R_n(x)$ is the *remainder,* or *error term.* It's the difference between the degree n Taylor polynomial at x, which is just an approximation, and the exact value $f(x)$. Notice that it depends on some unidentified number c that lies somewhere between a and x.

This error term is sometimes called the Lagrange Error. See examples on the next page.

Remainder estimation

1. **Let $f(x) = \sqrt{x+1}$. Compute the Taylor series around $a = 0$ up to x^2 and use this to estimate $\sqrt{1.1}$. What is the error?**

   We differentiate:

   - $f(x) = (x+1)^{1/2}$ so $f(0) = 1$
   - $f'(x) = \frac{1}{2}(x+1)^{-1/2}$ so $f'(0) = \frac{1}{2}$
   - $f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)(x+1)^{-3/2}$ so $f''(0) = -\frac{1}{4}$
   - $f'''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(x+1)^{-5/2} = \frac{3}{8(\sqrt{x+1})^5}$

   Thus the Taylor polynomial of degree 2 around $a = 0$ is:

   $$P_2(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2$$

   We note that we are trying to estimate $\sqrt{1.1}$, which it $f(0.1)$, so we plug in $P_2(0.1)$ to get our estimate:

   $$P_2(0.1) = 1 + \frac{1}{20} - \frac{1}{800} = \frac{839}{800}$$

   Now we look for the error. When we go up to degree 2, we have a formula for the error: $E = \frac{f^{(3)}(c)}{3!}(0.1)^3$ for some c between the centre and the point we're estimating at–i.e., $0 < c < 0.1$.

   Check Arithmetic

   So

   $E = \frac{3}{3! \cdot 8(\sqrt{c+1})^5} \frac{1}{1000} = \frac{3}{64000}(\sqrt{c+1})^5$.

   And we know c is between 0 and 0.1, and we want to know how large the error could possibly be. We see that it is largest when the denominator is smalllest, so it is largest when $c = 0$, and thus $E < \frac{3}{64000}$. **1/16000**

2. **Let $f(x) = \sqrt{x+1}$. Compute the Taylor series around $a = 3$ up to the degree-2 term and use this to estimate $\sqrt{4.1}$. What is the error?**

   We differentiate:

   - $f(x) = (x+1)^{1/2}$ so $f(3) = 2$
   - $f'(x) = \frac{1}{2}(x+1)^{-1/2}$ so $f'(3) = \frac{1}{4}$
   - $f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)(x+1)^{-3/2}$ so $f''(0) = -\frac{1}{32}$
   - $f'''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(x+1)^{-5/2} = \frac{3}{8(\sqrt{x+1})^5}$

   Thus the Taylor polynomial of degree 2 around $a = 0$ is:

   $$P_2(x) = 2 + \frac{1}{4}(x-3) - \frac{1}{32}(x-3)^2$$

We note that we are trying to estimate $\sqrt{4.1}$, which it $f(3.1)$, so we plug in $P_2(3.1)$ to get our estimate:

$$P_2(3.1) = 2 + \frac{1}{40} - \frac{1}{3200} = \frac{6479}{3200}$$

Now we look for the error. When we go up to degree 2, we have a formula for the error:

$E = \frac{f^{(3)}(c)}{3!}(3.1-3)^3$ for some c between the centre and the point we're estimating at–i.e., $3 < c < 3.1$. So

$E = \frac{3}{3!\cdot 8(\sqrt{c+1})^5}\frac{1}{1000} = \frac{3}{64000}(\sqrt{c+1})^5$.

And we know c is between 3 and 3.1, and we want to know how large the error could possibly be. We see that it is largest when the denominator is smalllest, so it is largest when $c = 3$, and thus $E < \frac{3}{128000}$.

3. **Let $f(x) = \ln(x+5) - \ln 5$. Approximate this by $\frac{x}{5} - \frac{x^2}{50}$. What is the error in this estimate provided $|x| < 0.1$?**

   First, we compute the Taylor series for f around $a = 0$:

   $f(x) = \ln(x+5) - \ln(5)$, so $f(0) = 0$.

   $f'(x) = \frac{1}{x+5}$, so $f'(0) = \frac{1}{5}$.

   $f''(x) = -\frac{1}{(x+5)^2}$, so $f''(0) = -\frac{1}{25}$.

   $f'''(x) = \frac{2}{(x+5)^3}$.

   So we see that the given estimate is the degree-2 Taylor series for f, and therefore by Taylor's theorem, the absolute value of the error for $|x| < 0.1$ is exactly $|\frac{f^{(3)}(c)}{3!}x^3| = \frac{|x|^3}{3|c+5|^3}$. And $|x| < 0.1$ and c between 0 and x, so $-0.1 < x < 0.1$ and $-0.1 < c < 0.1$. So to make the error as big as possible, we make the numerator as big as possible, which happens when $x = 0.1$, and the denominator as small as possible, which happens when $c = -0.1$, so the error is, in absolute value, bounded by $\frac{(0.1)^3}{3(4.9)^3}$.

4. **Let $f(x) = \sin(x)$. Approximate this by $x - \frac{x^3}{6}$. What is the error in this estimate provided $|x| < 0.1$?**