Sneaking Up on $\frac{\sin 0}{0}$

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Problem

Since $\sin(0) = 0$, the expression $\frac{\sin x}{x}$ is *indeterminate* at $x = 0$. Graphing $\frac{\sin x}{x}$ shows that perhaps there is a removable discontinuity at $x = 0$. For now, our goal is to consider just the limit from the positive side by evaluating:

$$\lim_{\theta \to 0^+} \frac{\sin \theta}{\theta}$$

using the geometry of a sector of a unit circle in the first quadrant.

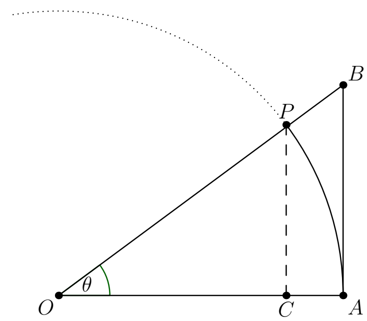

Figure 1: A first quadrant sector of a unit circle and triangle friends

Game Plan

The double inequality, $m \leq x \leq M$, puts lower and upper bounds on the value(s) of x. If we also know that $m = M = 1$, then we know x must equal 1.

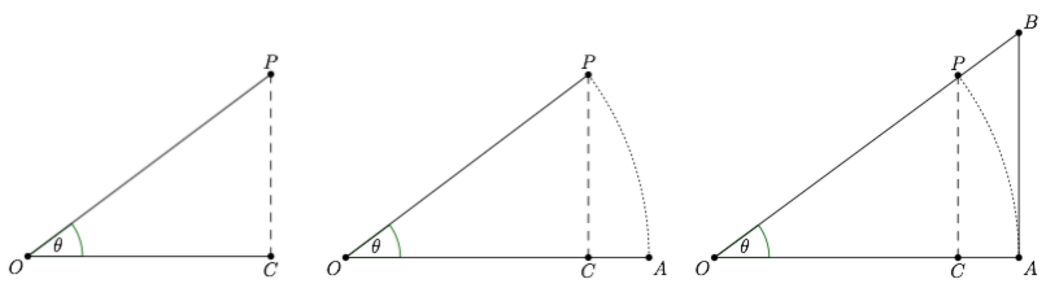

Figure 2: Comparing areas of the two triangles and sector from Figure 1

Looking at the sector of a unit circle in Figure 1, we see that sector AOP is part of $\triangle AOB$ and that $\triangle COP$ is part of sector AOP. This sets up a nice inequality[2]:

$$\alpha(\triangle COP) \leq \alpha(sector AOP) \leq \alpha(\triangle AOB)$$

Seeing that all three of these areas can be written in terms of θ, we can put lower and upper bounds on $\frac{\sin \theta}{\theta}$ for small positive values of *theta*. It turns out that both the upper and lower bounds both go to one as θ goes to zero in the first quadrant.

[1] This is sample problem write-up. Be sure to put your name, date, table number, and block number on your work.

[2] We use $\alpha(region)$ to mean the *area* of some region.

Solution

Consider each of the areas. We want to get expressions for each area in terms of just θ, $\sin\theta$ and $\cos\theta$.

$$\begin{aligned}
\alpha(\triangle COP) &= \frac{1}{2}\cdot OC\cdot PC = \frac{1}{2}\sin\theta\cos\theta \\
\alpha(sector AOP) &= \pi\cdot\frac{\theta}{2\pi} = \frac{1}{2}\cdot\theta \\
\alpha(\triangle AOB) &= \frac{1}{2}\cdot OA\cdot BA = \frac{1}{2}\cdot\tan\theta = \frac{1}{2}\cdot\frac{\sin\theta}{\cos\theta}
\end{aligned}$$

This sets up the inequality:

$$\frac{1}{2}\cdot\sin\theta\cos\theta \le \frac{1}{2}\cdot\theta \le \frac{1}{2}\cdot\frac{\sin\theta}{\cos\theta}$$

We clean up the inequality by gardening out $\frac{1}{2}$:

$$\sin\theta\cos\theta \le \theta \le \frac{\sin\theta}{\cos\theta}$$

Though we want $\dfrac{\sin\theta}{\theta}$ in the middle, its reciprocal, $\dfrac{\theta}{\sin\theta}$, is one move away and will get us close to where we want to be. We divide each expression by $\sin\theta$ and get:

$$\cos\theta \le \frac{\theta}{\sin\theta} \le \frac{1}{\cos\theta}$$

Now we are ready to take the limit of each expression as θ goes to zero from the positive side.

$$\lim_{\theta\to 0^+}\cos\theta \le \lim_{\theta\to 0^+}\frac{\theta}{\sin\theta} \le \lim_{\theta\to 0^+}\frac{1}{\cos\theta}$$

Since $\cos(\theta)$ is a continuous function,

$$\lim_{\theta\to 0}\cos\theta = \cos 0 = 1$$

We use this to evaluate the two outside limits.

$$\cos 0 \le \lim_{\theta\to 0^+}\frac{\theta}{\sin\theta} \le \frac{1}{\cos 0}$$

$$1 \le \lim_{\theta\to 0^+}\frac{\theta}{\sin\theta} \le 1$$

With lower and upper bounds of one,

$$\lim_{\theta\to 0^+}\frac{\theta}{\sin\theta} = 1$$

.
Since one is its own reciprocal,

$$\therefore \lim_{\theta\to 0^+}\frac{\sin\theta}{\theta} = 1$$

Final Thoughts

To really finish our evaluation of $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}$ we would need to also show,

$$\lim_{\theta\to 0^-}\frac{\sin\theta}{\theta} = 1$$

.
Perhaps we could use *symmetry* and the limit from the positive side to prove the limit from the negative side.