Revisiting Integrals

1. Assume $f(x)$ is a positive, invertible function on the interval $[a, b]$ where $b > a > 0$.

   (a) Sketch a possible graph of f on the domain $[a, b]$. Make sure your graph meets all of the criteria. On your graph, label a typical point (x, y) on the curve. Where is point $(x, f(x))$? Where is point $(f^{-1}(y), y)$? Explain.

   (b) Use your graph to show a geometric model of $A = \int_a^b f(x)\, dx$. Be sure to start with dA, a small piece of A, at your typical point (x, y) on your graph. Write a brief explanation.

   (c) Use your graph to show a geometric model of $B = \int_{f(a)}^{f(b)} f^{-1}(y)\, dy$. Start with dB, a small piece of B, at your typical point (x, y) on your graph. Write a brief explanation.

   (d) Use your graph to come up with a simple equation for the sum of the two integrals. [Hint: look for rectangles.]

   $$\int_a^b f(x)\, dx + \int_{f(a)}^{f(b)} f^{-1}(y)\, dy = ??$$

   (e) Solve your equation [from part (d)] for $\int_a^b f(x)\, dx$.

   (f) Use your answer from part (e) to evaluate $\int_1^e \ln(x)\, dx$

2. Now for a non-geometric look:

   (a) Use integration by parts to show:

   $$\int f(x)\,dx = xf(x) - \int xf'(x)\,dx$$

   (b) Use y-substitution to show:

   $$\int_a^b xf'(x)\,dx = \int_{f(a)}^{f(b)} f^{-1}(y)\,dy$$

   (c) Use parts (a) and (b) to show your result from question 1 part (e) is correct.

3. Summarize what you have learned from this assignment in your journal. Include equations and pictures.