Properties of Definite Integrals

1. $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
2. $\int_a^b k\cdot f(x)\,dx = k\int_a^b f(x)\,dx$
3. $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
4. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
5. If $g(x) \geq f(x)$ for all $x \in [a,b]$ then $\int_a^b g(x)\,dx \geq \int_a^b f(x)\,dx$

Questions

1. Briefly explain why each of the five properties of definite integrals is true.
2. Use the properties of definite integrals to show:

   If $m \leq f(x) \leq M$ for all $x \in [a,b]$ then

   $$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

3. Sketch the graph of a function f with the property that:

   $$\left| \int_1^5 f(x)\,dx \right| < \int_1^5 |f(x)|\,dx$$

4. Show that $\int_0^{\pi/2} \sqrt{1+\cos(2x)}\,dx = \sqrt{2}\int_0^{\pi/2} \cos x\,dx$
5. Show that $\frac{\pi}{2} < \int_0^{\pi} \cos(\sin x)\,dx \leq \pi$
   (Without using a calculator.)
6. Consider some function $f(x)$ where $f''(x)$ is continuous on the closed interval $[2,5]$, $f''(3) = 0$, and $f''(x) \neq 0$ for $x \in [2,3)$ and $x \in (3,5]$.
   (a) Sketch some possible graphs for $f''(x)$.
   (b) Is it possible that $(3, f(3))$ is not an *inflection point* for $f(x)$? If so, give an example by drawing a picture. If not, explain why.
7. Given $f'(x) = \cos(x)$,
   (a) Find a possible equation for $f(x)$.
   (b) Are there any other correct answers part (a)? Explain.