# *Continuity in More Detail*

## *5-Minute Review: Continuity*

We have worked off and on with continuous functions. Recall

**DEFINITION 8.1** (Continuity at a Point). A function $f(x)$ is **continuous at a point** $a$ if $\lim_{x \to a} f(x) = f(a)$. If $f$ is not continuous at $a$, then $a$ is a point of **discontinuity**.

Remember that this definition presumes that $f(a)$ is defined (i.e., $a$ is in the domain of $f$) and that $\lim_{x \to a} f(x)$ exists. To verify that a function is actually continuous at a point $a$, it is helpful to use the following checklist:

*Continuity Checklist.* A function $f$ is continuous at $a$ if the following three conditions hold:

1. $f(a)$ is defined (i.e., $a$ is in the domain of $f$).
2. $\lim_{x \to a} f(x)$ exists.
3. $\lim_{x \to a} f(x) = f(a)$.

**EXAMPLE 8.1.** Let $p(x) = 2x^2 + 5x - 11$. Then since $p$ is a polynomial, we know that $\lim_{x \to a} p(x) = p(a)$.

## *Functions We Know Are Continuous*

In our study of limits we remarked that polynomials are continuous everywhere since $\lim_{x \to a} p(x) = p(a)$ for any polynomial $p$ and any number $a$. Using the quotient rule for limits, we then saw that a rational function $r$ is continuous at each point in its domain, since $\lim_{x \to a} r(x) = r(a)$ for any $a$ at which $r$ is defined.

**THEOREM 8.1.** (Polynomials and Rational Functions Are Continuous) The following types of functions are continuous.

1. A polynomial is continuous everywhere, i.e. for all $x$.
2. A rational functions $r(x) = \frac{p(x)}{q(x)}$ where $p$ and $q$ are polynomials is continuous at all points in its domain, i.e., where $q(x) \neq 0$.

Take a moment to write down other types of functions that you know are continuous on their domains. We we see several more 'types' of functions that are continuous in the coming days. Knowing that a function is continuous makes limit calculations trivial: If $f(x)$ is continuous at $a$, to evaluate $\lim_{x \to a} f(x)$ all we need to do is evaluate $f$ at $a$, that is, $\lim_{x \to a} f(x) = f(a)$. No other work is required. This is what makes them important. There are no surprises!

## *Determining Where Functions Are Continuous*

**EXAMPLE 8.2.** Determine whether the following functions are continuous at the given points.

(*a*) $r(x) = \dfrac{x^2-1}{x^2-4x+3}$ at $a = 1, 2,$ and 3.

(*b*) $g(x) = \begin{cases} x+2, \text{if } x \geq 3 \\ x^2+1, \text{if } x < 3 \end{cases}$ at $a = 0$ and 3.

**SOLUTION.** (*a*) $r(x)$ is a rational function. A rational function is continuous at every point in its domain. Notice that

$$r(x) = \frac{x^2-1}{x^2-4x+3} = \frac{(x-1)(x+1)}{(x-1)(x-3)}$$

is not defined at $x = 1$ and $x = 3$, so $r$ is not continuous at either of these points. However, since $x = 2$ is in the domain of $r$, then $r$ is continuous at $x = 2$ and, in fact, at every real number not equal to 1 or 3. In fact it is easy to calculate

$$\lim_{x\to 2} \frac{x^2-1}{x^2-4x+3} = \frac{4-1}{4-8+3} = -3 = f(2).$$

(*b*) $g(x)$ is a piecewise function and the definition of $g$ changes at $x = 3$. Use the checklist.

(1) $g(3) = 3+2 = 5$.

(2) To determine $\lim_{x\to 3} g(x)$ use one-sided limits since the definition of $g$ is different on either side of 3.

$$\lim_{x\to 3^+} g(x) = \lim_{x\to 3^+} x+2 = 5$$

while

$$\lim_{x\to 3^-} g(x) = \lim_{x\to 3^-} 3^2+1 = 10.$$

Since the two one-sided limits differ, $\lim_{x\to 3} g(x)$ DNE. So $g$ is not continuous at $x = 3$.

(*c*) What about $g$ at $x = 0$? Since 0 is less than 3, $g(0) = 0^2+1 = 1$. Further, near 0 on either side, $g(x) = x^2+1$ so $\lim_{x\to 0} g(x) = \lim_{x\to 0} x^2+1 = 1 = g(0)$. So $g$ is continuous at $x = 0$.

**EXAMPLE 8.3.** Let $g(x) = \begin{cases} x^2+m, \text{if } x \leq 2 \\ mx+7, \text{if } x > 2 \end{cases}$, where $m$ is a constant. Is there any value of $m$ for which $g$ would be continuous at $x = 2$?

**SOLUTION.** $g(x)$ is a piecewise function and the definition of $g$ changes at $x = 2$.

(1) $g(2) = 2^2+m$.

(2) To find $\lim_{x\to 2} g(x)$ we must use one-sided limits since the definition of $g$ is different on either side of 2.

$$\lim_{x\to 2^+} g(x) = \lim_{x\to 2^+} mx+7 = 2m+7$$

while

$$\lim_{x\to 2^-} g(x) = \lim_{x\to 2^-} x^2+m = 4+m.$$

We need the two one-sided limits to be equal: $2m+7 = 4+m \Rightarrow m = -3$. If $m = -3$, then

$$\lim_{x\to 2^+} g(x) = \lim_{x\to 2^+} -3x+7 = 1$$

and

$$\lim_{x\to 2^-} g(x) = \lim_{x\to 2^-} x^2-3 = 1.$$

So $\lim_{x\to 2} g(x) = 1$.

(3) Now with $m = -3$, $g(2) = 2^2-3 = 1$ also. So $g$ is continuous at 2 if $m = -3$.

**EXAMPLE 8.4.** Let $f(x) = \begin{cases} 2x+4, \text{if } x < 1 \\ 5, \text{if } x = 1 \\ -x+7, \text{if } x > 1 \end{cases}$. Is $f$ continuous at $x = 0$? At $x = 1$?

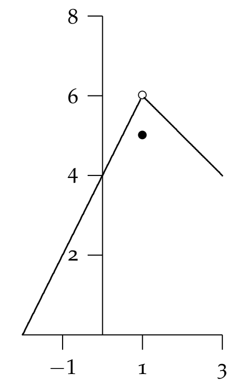

Figure 8.1: $f$ is *not* continuous at $x = 1$. It has a *removable discontinuity* there. See Definition 8.2.

**SOLUTION.** At $x = 0$: Use the continuity checklist.

(1) Since $0 < 1$, we have $f(0) = 2(0) + 4 = 4$.

(2) $\lim_{x\to 0} f(x) \overset{x \leq 1}{=} \lim_{x\to 0} 2x + 4 = 4$,

(3) So $f(0) = \lim_{x\to 0} f(x)$ and $f$ is continuous at 0

At $x = 1$ the definition of $f$ changes.

(1) $f(1) = 5$.

(2) To find $\lim_{x\to 1} f(x)$ we must use one-sided limits since the definition of $f$ is different on either side of 1.

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} = -x + 7 = 6$$

while

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 2x + 4 = 6.$$

So $\lim_{x\to 1} f(x) = 6$.

(3) Finally, $f(1) \neq \lim_{x\to 1} f(x)$ also. So $f$ is not continuous at 1.

## *Removable Discontinuities*

In Example 8.4 just above, even though $f$ was not continuous at $x = 1$, the behavior of $f$ *near* 1 was reasonable. The limit existed (and was finite). The problem was that the limit value and the function value were different. We give such points a special name.

**DEFINITION 8.2.** (Removable Discontinuity) A function $f$ has a **removable discontinuity** (RD) at $a$ if the following hold:

**1.** $\lim_{x\to a} f(x)$ exists (and is finite).

**2.** $\lim_{x\to a} f(x) \neq f(a)$. Note: $f(a)$ may not even exist.

Remember that $f$ is continuous at $a$ if $\lim_{x\to a} f(x) = f(a)$. So condition 2 in the definition ensures that $f$ is NOT continuous at $a$. On the other hand, the function is well-behaved near $a$, since $\lim_{x\to a} f(x)$ exists. In fact, if we defined (or redefined) $f(a)$ to be $\lim_{x\to a} f(x)$, then $f$ would be continuous. That is, we could *remove the discontinuity* by redefining $f$ and filling in the hole in the graph (see Figure 8.1).

The next example is more typical of where we see removable discontinuities.

**EXAMPLE 8.5.** Determine the points at which $f(x) = \dfrac{x^2 - 5x + 6}{x^3 + x^2 - 12x}$ is discontinuous. At which points does $f$ have VA's? Removable discontinuities?

**SOLUTION.** Since $f(x)$ is rational it is continuous at all points in its domain. So it will fail to be continuous where the denominator is equal to 0. So let's factor $f$:

$$f(x) = \frac{x^2 - 5x + 6}{x^3 + x^2 - 12x} = \frac{(x-2)(x-3)}{x(x+4)(x-3)}, \qquad x \neq -4, 0, 3.$$

$f$ is discontinuous at $-4$, 0, and 3. Now examine appropriate limits to check for VA's and removable discontinuities. (Can you predict which are which?)

At $x = -4$:

$$\lim_{x\to-4^-} f(x) = \lim_{x\to-4^-} \frac{(x-2)(x-3)}{x(x+4)(x-3)} = \lim_{x\to-4^-} \frac{\overbrace{x-2}^{\to-6}}{\underbrace{x(x+4)}_{\to-4\cdot 0^-=0^+}} = -\infty.$$

This is enough to conclude that $f$ has a VA at $-4$. Caution: Take care with the calculation of the sign in the denominator.

At $x = 0$:

$$\lim_{x\to0^-} f(x) = \lim_{x\to0^-} \frac{(x-2)(x-3)}{x(x+4)(x-3)} = \lim_{x\to0^-} \frac{\overbrace{x-2}^{\to-2}}{\underbrace{x(x+4)}_{\to0^-\cdot4=0^-}} = -\infty.$$

This is enough to conclude that $f$ has a VA at 0. Again: Take care with the calculation of the sign in the denominator.

At $x = 3$: Having seen the factorization of $f$, we know that we can calculate a two-sided limit at 3.

$$\lim_{x\to3} f(x) = \lim_{x\to3} \frac{(x-2)(x-3)}{x(x+4)(x-3)} = \lim_{x\to3} \frac{x-2}{x(x+4)} = \frac{1}{21}.$$

Since $\lim_{x\to3} f(x)$ exists but $f(3)$ is not defined, then $f$ has a removable discontinuity at $x = 3$.

**YOU TRY IT 8.1.** Determine $\lim_{x\to-4^+} f(x)$ and $\lim_{x\to0^+} f(x)$ for the function in Example 8.5.

Answer to YOU TRY IT 8.1 : Both are $\infty$.

**EXAMPLE 8.6.** Determine the points at which $f(x) = \dfrac{\frac{1}{x-2} - \frac{1}{2}}{x-4}$ is discontinuous. At which points does $f$ have VA's? Removable discontinuities?

**SOLUTION.** Since $f(x)$ is rational it is continuous at all points in its domain. We can see immediately that $f$ is not defined at $x = 4$ and $x = 2$, where there would be division by 0, so $f$ is not continuous at these two points. Let's simplify the expression for $f$ before taking the appropriate limits.

$$f(x) = \frac{\frac{1}{x-2} - \frac{1}{2}}{x-4} = \frac{\frac{2-(x-2)}{2(x-2)}}{x-4} = \frac{4-x}{2(x-2)(x-4)}.$$

At $x = 2$:

$$\lim_{x\to2^-} f(x) = \lim_{x\to2^-} \frac{4-x}{2(x-2)(x-4)} = \lim_{x\to2^-} \frac{-1}{2(x-2)} = \lim_{x\to2^-} \frac{-1}{\underbrace{2(x-2)}_{\to2\cdot0^-=0^-}} = \infty.$$

This is enough to conclude that $f$ has a VA at 2.

At $x = 4$: Having seen the factorization of $f$, we know that we can calculate a two-sided limit at 4.

$$\lim_{x\to4} f(x) = \lim_{x\to4} \frac{4-x}{2(x-2)(x-4)} = \lim_{x\to4} \frac{-1}{2(x-2)} = -\frac{1}{8}.$$

Since $\lim_{x\to4} f(x)$ exists but $f(4)$ is not defined, then $f$ has a removable discontinuity at $x = 4$.

**YOU TRY IT 8.2.** Determine $\lim_{x\to2^+} f(x)$ for the function in Example 8.6.

Answer to YOU TRY IT 8.2 : $-\infty$.

**YOU TRY IT 8.3.** Determine where the function $f(x) = \dfrac{x^2-1}{x^2-3x+2}$ has vertical asymptotes and where it has removable discontinuities.

Answer to YOU TRY IT 8.3 : VA at $x = 2$. Removable discontinuity at $x = 1$ since $\lim_{x\to1} f(x) = -2$ and $f(a)$ UND.

**YOU TRY IT 8.4.** Consider the two graphs below that we saw earlier this term when first considering limits. Discuss the type of discontinuity in each.

Answer to YOU TRY IT 8.4 : Since $\lim_{x\to0} \frac{\sin x}{x} = 1$ but $\frac{\sin x}{x}$ is not defined at 0, then by Definition **??** there is a removable discontinuity at 0.

On the right $\lim_{x\to2} f(x)$ DNE because the two one-sided derivatives are not equal. So the discontinuity at 2 is not removable.

## *8.0 Problems*