3

Rules for Finding Derivatives

It is tedious to compute a limit every time we need to know the derivative of a function. Fortunately, we can develop a small collection of examples and rules that allow us to compute the derivative of almost any function we are likely to encounter. Many functions involve quantities raised to a constant power, such as polynomials and more complicated combinations like $y = (\sin x)^4$. So we start by examining powers of a single variable; this gives us a building block for more complicated examples.

3.1 The Power Rule

We start with the derivative of a power function, $f(x) = x^n$. Here n is a number of any kind: integer, rational, positive, negative, even irrational, as in x^π. We have already computed some simple examples, so the formula should not be a complete surprise:

$$\frac{d}{dx}x^n = nx^{n-1}.$$

It is not easy to show this is true for any n. We will do some of the easier cases now, and discuss the rest later.

The easiest, and most common, is the case that n is a positive integer. To compute the derivative we need to compute the following limit:

$$\frac{d}{dx}x^n = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^n - x^n}{\Delta x}.$$

For a specific, fairly small value of n, we could do this by straightforward algebra.

EXAMPLE 3.1.1 Find the derivative of $f(x) = x^3$.

$$\begin{aligned}
\frac{d}{dx}x^3 &= \lim_{\Delta x\to 0}\frac{(x+\Delta x)^3 - x^3}{\Delta x}.\\
&= \lim_{\Delta x\to 0}\frac{x^3+3x^2\Delta x+3x\Delta x^2+\Delta x^3-x^3}{\Delta x}.\\
&= \lim_{\Delta x\to 0}\frac{3x^2\Delta x+3x\Delta x^2+\Delta x^3}{\Delta x}.\\
&= \lim_{\Delta x\to 0} 3x^2+3x\Delta x+\Delta x^2 = 3x^2.
\end{aligned}$$

□

The general case is really not much harder as long as we don't try to do too much. The key is understanding what happens when $(x + \Delta x)^n$ is multiplied out:

$$(x+\Delta x)^n = x^n + nx^{n-1}\Delta x + a_2x^{n-2}\Delta x^2 + \cdots + +a_{n-1}x\Delta x^{n-1} + \Delta x^n.$$

We know that multiplying out will give a large number of terms all of the form $x^i\Delta x^j$, and in fact that $i+j=n$ in every term. One way to see this is to understand that one method for multiplying out $(x+\Delta x)^n$ is the following: In every $(x+\Delta x)$ factor, pick either the x or the Δx, then multiply the n choices together; do this in all possible ways. For example, for $(x+\Delta x)^3$, there are eight possible ways to do this:

$$\begin{aligned}
(x+\Delta x)(x+\Delta x)(x+\Delta x) &= xxx + xx\Delta x + x\Delta x x + x\Delta x\Delta x\\
&\quad + \Delta x xx + \Delta x x\Delta x + \Delta x\Delta x x + \Delta x\Delta x\Delta x\\
&= x^3 + x^2\Delta x + x^2\Delta x + x\Delta x^2\\
&\quad + x^2\Delta x + x\Delta x^2 + x\Delta x^2 + \Delta x^3\\
&= x^3 + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3
\end{aligned}$$

No matter what n is, there are n ways to pick Δx in one factor and x in the remaining $n-1$ factors; this means one term is $nx^{n-1}\Delta x$. The other coefficients are somewhat harder to understand, but we don't really need them, so in the formula above they have simply been called a_2, a_3, and so on. We know that every one of these terms contains Δx to at least the power 2. Now let's look at the limit:

$$\begin{aligned}
\frac{d}{dx}x^n &= \lim_{\Delta x\to 0}\frac{(x+\Delta x)^n - x^n}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{x^n + nx^{n-1}\Delta x + a_2x^{n-2}\Delta x^2 + \cdots + a_{n-1}x\Delta x^{n-1} + \Delta x^n - x^n}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{nx^{n-1}\Delta x + a_2x^{n-2}\Delta x^2 + \cdots + a_{n-1}x\Delta x^{n-1} + \Delta x^n}{\Delta x}\\
&= \lim_{\Delta x\to 0} nx^{n-1} + a_2x^{n-2}\Delta x + \cdots + a_{n-1}x\Delta x^{n-2} + \Delta x^{n-1} = nx^{n-1}.
\end{aligned}$$

Now without much trouble we can verify the formula for negative integers. First let's look at an example:

EXAMPLE 3.1.2 Find the derivative of $y = x^{-3}$. Using the formula, $y' = -3x^{-3-1} = -3x^{-4}$. □

Here is the general computation. Suppose n is a negative integer; the algebra is easier to follow if we use $n = -m$ in the computation, where m is a positive integer.

$$\begin{aligned}
\frac{d}{dx}x^n = \frac{d}{dx}x^{-m} &= \lim_{\Delta x\to 0}\frac{(x+\Delta x)^{-m} - x^{-m}}{\Delta x} \\
&= \lim_{\Delta x\to 0}\frac{\frac{1}{(x+\Delta x)^m} - \frac{1}{x^m}}{\Delta x} \\
&= \lim_{\Delta x\to 0}\frac{x^m - (x+\Delta x)^m}{(x+\Delta x)^m x^m \Delta x} \\
&= \lim_{\Delta x\to 0}\frac{x^m - (x^m + mx^{m-1}\Delta x + a_2x^{m-2}\Delta x^2 + \cdots + a_{m-1}x\Delta x^{m-1} + \Delta x^m)}{(x+\Delta x)^m x^m \Delta x} \\
&= \lim_{\Delta x\to 0}\frac{-mx^{m-1} - a_2x^{m-2}\Delta x - \cdots - a_{m-1}x\Delta x^{m-2} - \Delta x^{m-1})}{(x+\Delta x)^m x^m} \\
&= \frac{-mx^{m-1}}{x^m x^m} = \frac{-mx^{m-1}}{x^{2m}} = -mx^{m-1-2m} = nx^{-m-1} = nx^{n-1}.
\end{aligned}$$

We will later see why the other cases of the power rule work, but from now on we will use the power rule whenever n is any real number. Let's note here a simple case in which the power rule applies, or almost applies, but is not really needed. Suppose that $f(x) = 1$; remember that this "1" is a function, not "merely" a number, and that $f(x) = 1$ has a graph that is a horizontal line, with slope zero everywhere. So we know that $f'(x) = 0$. We might also write $f(x) = x^0$, though there is some question about just what this means at $x = 0$. If we apply the power rule, we get $f'(x) = 0x^{-1} = 0/x = 0$, again noting that there is a problem at $x = 0$. So the power rule "works" in this case, but it's really best to just remember that the derivative of any constant function is zero.

Exercises 3.1.

Find the derivatives of the given functions.

1. x^{100} ⇒

2. x^{-100} ⇒

3. $\dfrac{1}{x^5}$ ⇒

4. x^{π} ⇒

5. $x^{3/4}$ ⇒

6. $x^{-9/7}$ ⇒

3.2 Linearity of the Derivative

An operation is linear if it behaves "nicely" with respect to multiplication by a constant and addition. The name comes from the equation of a line through the origin, $f(x) = mx$, and the following two properties of this equation. First, $f(cx) = m(cx) = c(mx) = cf(x)$, so the constant c can be "moved outside" or "moved through" the function f. Second, $f(x + y) = m(x + y) = mx + my = f(x) + f(y)$, so the addition symbol likewise can be moved through the function.

The corresponding properties for the derivative are:

$$(cf(x))' = \frac{d}{dx}cf(x) = c\frac{d}{dx}f(x) = cf'(x),$$

and

$$(f(x) + g(x))' = \frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x) = f'(x) + g'(x).$$

It is easy to see, or at least to believe, that these are true by thinking of the distance/speed interpretation of derivatives. If one object is at position $f(t)$ at time t, we know its speed is given by $f'(t)$. Suppose another object is at position $5f(t)$ at time t, namely, that it is always 5 times as far along the route as the first object. Then it "must" be going 5 times as fast at all times.

The second rule is somewhat more complicated, but here is one way to picture it. Suppose a flat bed railroad car is at position $f(t)$ at time t, so the car is traveling at a speed of $f'(t)$ (to be specific, let's say that $f(t)$ gives the position on the track of the rear end of the car). Suppose that an ant is crawling from the back of the car to the front so that its position *on the car* is $g(t)$ and its speed *relative to the car* is $g'(t)$. Then in reality, at time t, the ant is at position $f(t) + g(t)$ along the track, and its speed is "obviously" $f'(t) + g'(t)$.

We don't want to rely on some more-or-less obvious physical interpretation to determine what is true mathematically, so let's see how to verify these rules by computation.

We'll do one and leave the other for the exercises.

$$\begin{aligned}
\frac{d}{dx}(f(x)+g(x)) &= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)+g(x+\Delta x)-(f(x)+g(x))}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)+g(x+\Delta x)-f(x)-g(x)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)+g(x+\Delta x)-g(x)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\left(\frac{f(x+\Delta x)-f(x)}{\Delta x}+\frac{g(x+\Delta x)-g(x)}{\Delta x}\right)\\
&= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}+\lim_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{\Delta x}\\
&= f'(x)+g'(x)
\end{aligned}$$

This is sometimes called the **sum rule** for derivatives.

EXAMPLE 3.2.1 Find the derivative of $f(x)=x^5+5x^2$. We have to invoke linearity twice here:

$$f'(x)=\frac{d}{dx}(x^5+5x^2)=\frac{d}{dx}x^5+\frac{d}{dx}(5x^2)=5x^4+5\frac{d}{dx}(x^2)=5x^4+5\cdot 2x^1=5x^4+10x.$$

□

Because it is so easy with a little practice, we can usually combine all uses of linearity into a single step. The following example shows an acceptably detailed computation.

EXAMPLE 3.2.2 Find the derivative of $f(x)=3/x^4-2x^2+6x-7$.

$$f'(x)=\frac{d}{dx}\left(\frac{3}{x^4}-2x^2+6x-7\right)=\frac{d}{dx}(3x^{-4}-2x^2+6x-7)=-12x^{-5}-4x+6.$$

□

Exercises 3.2.

Find the derivatives of the functions in 1–6.

1. $5x^3+12x^2-15$ ⇒

2. $-4x^5+3x^2-5/x^2$ ⇒

3. $5(-3x^2+5x+1)$ ⇒

4. $f(x)+g(x)$, where $f(x)=x^2-3x+2$ and $g(x)=2x^3-5x$ ⇒

5. $(x+1)(x^2+2x-3)$ ⇒

6. $\sqrt{625-x^2}+3x^3+12$ (See section 2.1.) ⇒

7. Find an equation for the tangent line to $f(x)=x^3/4-1/x$ at $x=-2$. ⇒

8. Find an equation for the tangent line to $f(x) = 3x^2 - \pi^3$ at $x = 4$. ⇒

9. Suppose the position of an object at time t is given by $f(t) = -49t^2/10 + 5t + 10$. Find a function giving the speed of the object at time t. The acceleration of an object is the rate at which its speed is changing, which means it is given by the derivative of the speed function. Find the acceleration of the object at time t. ⇒

10. Let $f(x) = x^3$ and $c = 3$. Sketch the graphs of f, cf, f', and $(cf)'$ on the same diagram.

11. The general polynomial P of degree n in the variable x has the form $P(x) = \sum_{k=0}^{n} a_k x^k = a_0 + a_1 x + \ldots + a_n x^n$. What is the derivative (with respect to x) of P? ⇒

12. Find a cubic polynomial whose graph has horizontal tangents at $(-2, 5)$ and $(2, 3)$. ⇒

13. Prove that $\frac{d}{dx}(cf(x)) = cf'(x)$ using the definition of the derivative.

14. Suppose that f and g are differentiable at x. Show that $f - g$ is differentiable at x using the two linearity properties from this section.