10.5 Calculus with Parametric Equations

We have already seen how to compute slopes of curves given by parametric equations—it is how we computed slopes in polar coordinates.

EXAMPLE 10.5.1 Find the slope of the cycloid $x = t - \sin t$, $y = 1 - \cos t$. We compute $x' = 1 - \cos t$, $y' = \sin t$, so

$$\frac{dy}{dx} = \frac{\sin t}{1 - \cos t}.$$

Note that when t is an odd multiple of π, like π or 3π, this is $(0/2) = 0$, so there is a horizontal tangent line, in agreement with figure 10.4.1. At even multiples of π, the fraction is $0/0$, which is undefined. The figure shows that there is no tangent line at such points. □

Areas can be a bit trickier with parametric equations, depending on the curve and the area desired. We can potentially compute areas between the curve and the x-axis quite easily.

EXAMPLE 10.5.2 Find the area under one arch of the cycloid $x = t - \sin t$, $y = 1 - \cos t$. We would like to compute

$$\int_0^{2\pi} y\, dx,$$

but we do not know y in terms of x. However, the parametric equations allow us to make a substitution: use $y = 1 - \cos t$ to replace y, and compute $dx = (1 - \cos t)\, dt$. Then the integral becomes

$$\int_0^{2\pi} (1 - \cos t)(1 - \cos t)\, dt = 3\pi.$$

Note that we need to convert the original x limits to t limits using $x = t - \sin t$. When $x = 0$, $t = \sin t$, which happens only when $t = 0$. Likewise, when $x = 2\pi$, $t - 2\pi = \sin t$ and $t = 2\pi$. Alternately, because we understand how the cycloid is produced, we can see directly that one arch is generated by $0 \le t \le 2\pi$. In general, of course, the t limits will be different than the x limits. □

This technique will allow us to compute some quite interesting areas, as illustrated by the exercises.

As a final example, we see how to compute the length of a curve given by parametric equations. Section 9.9 investigates arc length for functions given as y in terms of x, and develops the formula for length:

$$\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

Using some properties of derivatives, including the chain rule, we can convert this to use parametric equations $x = f(t)$, $y = g(t)$:

$$\begin{aligned}\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2\left(\frac{dy}{dx}\right)^2}\;\frac{dt}{dx}\,dx\\ &= \int_u^v \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt\\ &= \int_u^v \sqrt{(f'(t))^2 + (g'(t))^2}\,dt.\end{aligned}$$

Here u and v are the t limits corresponding to the x limits a and b.

EXAMPLE 10.5.3 Find the length of one arch of the cycloid. From $x = t - \sin t$, $y = 1 - \cos t$, we get the derivatives $f' = 1 - \cos t$ and $g' = \sin t$, so the length is

$$\int_0^{2\pi} \sqrt{(1-\cos t)^2 + \sin^2 t}\,dt = \int_0^{2\pi} \sqrt{2 - 2\cos t}\,dt.$$

Now we use the formula $\sin^2(t/2) = (1-\cos(t))/2$ or $4\sin^2(t/2) = 2 - 2\cos t$ to get

$$\int_0^{2\pi} \sqrt{4\sin^2(t/2)}\,dt.$$

Since $0 \le t \le 2\pi$, $\sin(t/2) \ge 0$, so we can rewrite this as

$$\int_0^{2\pi} 2\sin(t/2)\,dt = 8.$$

□