10.3 Areas in polar coordinates

We can use the equation of a curve in polar coordinates to compute some areas bounded by such curves. The basic approach is the same as with any application of integration: find an approximation that approaches the true value. For areas in rectangular coordinates, we approximated the region using rectangles; in polar coordinates, we use sectors of circles, as depicted in figure 10.3.1. Recall that the area of a sector of a circle is $\alpha r^2/2$, where α is the angle subtended by the sector. If the curve is given by $r = f(\theta)$, and the angle subtended by a small sector is $\Delta\theta$, the area is $(\Delta\theta)(f(\theta))^2/2$. Thus we approximate the total area as

$$\sum_{i=0}^{n-1} \frac{1}{2} f(\theta_i)^2 \, \Delta\theta.$$

In the limit this becomes

$$\int_a^b \frac{1}{2} f(\theta)^2 \, d\theta.$$

EXAMPLE 10.3.1 We find the area inside the cardioid $r = 1 + \cos\theta$.

$$\int_0^{2\pi} \frac{1}{2}(1+\cos\theta)^2 \, d\theta = \frac{1}{2}\int_0^{2\pi} 1+2\cos\theta+\cos^2\theta \, d\theta = \frac{1}{2}\left(\theta + 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)\Big|_0^{2\pi} = \frac{3\pi}{2}.$$

□

EXAMPLE 10.3.2 We find the area between the circles $r = 2$ and $r = 4\sin\theta$, as shown in figure 10.3.2. The two curves intersect where $2 = 4\sin\theta$, or $\sin\theta = 1/2$, so $\theta = \pi/6$ or

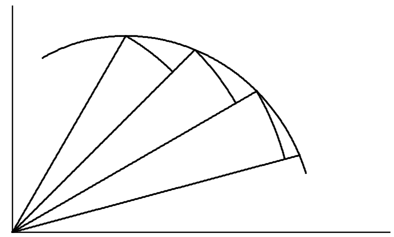

Figure 10.3.1 Approximating area by sectors of circles.

$5\pi/6$. The area we want is then

$$\frac{1}{2}\int_{\pi/6}^{5\pi/6} 16\sin^2\theta - 4\,d\theta = \frac{4}{3}\pi + 2\sqrt{3}.$$

□

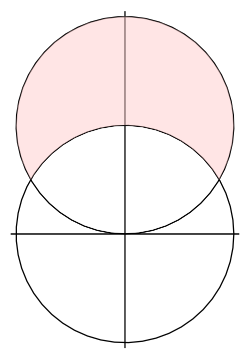

Figure 10.3.2 An area between curves.

This example makes the process appear more straightforward than it is. Because points have many different representations in polar coordinates, it is not always so easy to identify points of intersection.

EXAMPLE 10.3.3 We find the shaded area in the first graph of figure 10.3.3 as the difference of the other two shaded areas. The cardioid is $r = 1 + \sin\theta$ and the circle is $r = 3\sin\theta$. We attempt to find points of intersection:

$$\begin{aligned} 1 + \sin\theta &= 3\sin\theta \\ 1 &= 2\sin\theta \\ 1/2 &= \sin\theta. \end{aligned}$$

This has solutions $\theta = \pi/6$ and $5\pi/6$; $\pi/6$ corresponds to the intersection in the first quadrant that we need. Note that no solution of this equation corresponds to the intersection point at the origin, but fortunately that one is obvious. The cardioid goes through the origin when $\theta = -\pi/2$; the circle goes through the origin at multiples of π, starting with 0.

Now the larger region has area

$$\frac{1}{2}\int_{-\pi/2}^{\pi/6} (1+\sin\theta)^2\,d\theta = \frac{\pi}{2} - \frac{9}{16}\sqrt{3}$$

and the smaller has area

$$\frac{1}{2}\int_{0}^{\pi/6} (3\sin\theta)^2\,d\theta = \frac{3\pi}{8} - \frac{9}{16}\sqrt{3}$$

so the area we seek is $\pi/8$. □

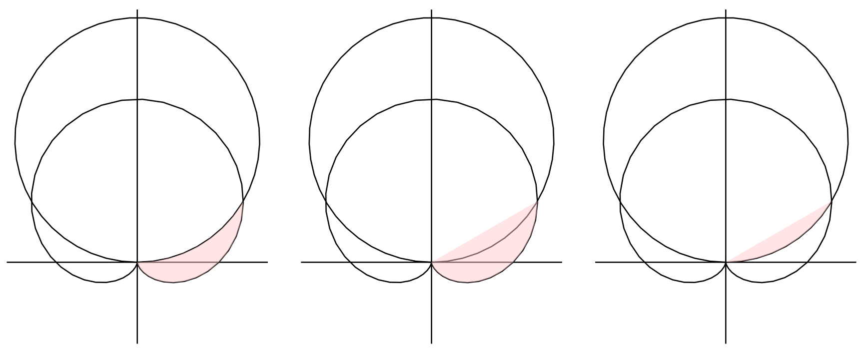

Figure 10.3.3 An area between curves.

Exercises 10.3.

Find the area enclosed by the curve.

1. $r = \sqrt{\sin\theta}$ ⇒

2. $r = 2 + \cos\theta$ ⇒

3. $r = \sec\theta, \pi/6 \le \theta \le \pi/3$ ⇒

4. $r = \cos\theta, 0 \le \theta \le \pi/3$ ⇒

5. $r = 2a\cos\theta, a > 0$ ⇒

6. $r = 4 + 3\sin\theta$ ⇒

7. Find the area inside the loop formed by $r = \tan(\theta/2)$. ⇒

8. Find the area inside one loop of $r = \cos(3\theta)$. ⇒

9. Find the area inside one loop of $r = \sin^2\theta$. ⇒

10. Find the area inside the small loop of $r = (1/2) + \cos\theta$. ⇒

11. Find the area inside $r = (1/2) + \cos\theta$, including the area inside the small loop. ⇒

12. Find the area inside one loop of $r^2 = \cos(2\theta)$. ⇒

13. Find the area enclosed by $r = \tan\theta$ and $r = \dfrac{\csc\theta}{\sqrt{2}}$. ⇒

14. Find the area inside $r = 2\cos\theta$ and outside $r = 1$. ⇒

15. Find the area inside $r = 2\sin\theta$ and above the line $r = (3/2)\csc\theta$. ⇒

16. Find the area inside $r = \theta$, $0 \le \theta \le 2\pi$. ⇒

17. Find the area inside $r = \sqrt{\theta}$, $0 \le \theta \le 2\pi$. ⇒

18. Find the area inside both $r = \sqrt{3}\cos\theta$ and $r = \sin\theta$. ⇒

19. Find the area inside both $r = 1 - \cos\theta$ and $r = \cos\theta$. ⇒

20. The center of a circle of radius 1 is on the circumference of a circle of radius 2. Find the area of the region inside both circles. ⇒

21. Find the shaded area in figure 10.3.4. The curve is $r = \theta$, $0 \le \theta \le 3\pi$. ⇒

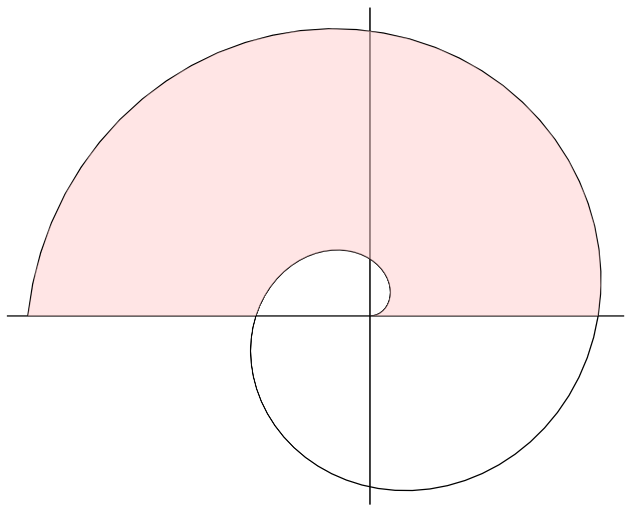

Figure 10.3.4 An area bounded by the spiral of Archimedes.