10.2 Slopes in polar coordinates

When we describe a curve using polar coordinates, it is still a curve in the x-y plane. We would like to be able to compute slopes and areas for these curves using polar coordinates.

We have seen that $x = r\cos\theta$ and $y = r\sin\theta$ describe the relationship between polar and rectangular coordinates. If in turn we are interested in a curve given by $r = f(\theta)$, then we can write $x = f(\theta)\cos\theta$ and $y = f(\theta)\sin\theta$, describing x and y in terms of θ alone. The first of these equations describes θ implicitly in terms of x, so using the chain rule we may compute

$$\frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx}.$$

Since $d\theta/dx = 1/(dx/d\theta)$, we can instead compute

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}.$$

EXAMPLE 10.2.1 Find the points at which the curve given by $r = 1 + \cos\theta$ has a vertical or horizontal tangent line. Since this function has period 2π, we may restrict our attention to the interval $[0, 2\pi)$ or $(-\pi, \pi]$, as convenience dictates. First, we compute the slope:

$$\frac{dy}{dx} = \frac{(1+\cos\theta)\cos\theta - \sin\theta\sin\theta}{-(1+\cos\theta)\sin\theta - \sin\theta\cos\theta} = \frac{\cos\theta + \cos^2\theta - \sin^2\theta}{-\sin\theta - 2\sin\theta\cos\theta}.$$

This fraction is zero when the numerator is zero (and the denominator is not zero). The numerator is $2\cos^2\theta + \cos\theta - 1$ so by the quadratic formula

$$\cos\theta = \frac{-1 \pm \sqrt{1 + 4\cdot 2}}{4} = -1 \quad \text{or} \quad \frac{1}{2}.$$

This means θ is π or $\pm\pi/3$. However, when $\theta = \pi$, the denominator is also 0, so we cannot conclude that the tangent line is horizontal.

Setting the denominator to zero we get

$$-\theta - 2\sin\theta\cos\theta = 0$$

$$\sin\theta(1 + 2\cos\theta) = 0,$$

so either $\sin\theta = 0$ or $\cos\theta = -1/2$. The first is true when θ is 0 or π, the second when θ is $2\pi/3$ or $4\pi/3$. However, as above, when $\theta = \pi$, the numerator is also 0, so we cannot conclude that the tangent line is vertical. Figure 10.2.1 shows points corresponding to θ equal to 0, $\pm\pi/3$, $2\pi/3$ and $4\pi/3$ on the graph of the function. Note that when $\theta = \pi$ the curve hits the origin and does not have a tangent line. □

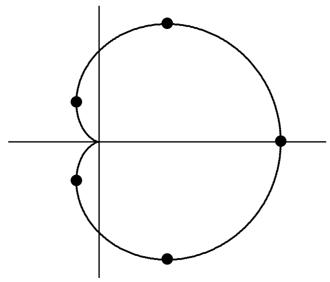

Figure 10.2.1 Points of vertical and horizontal tangency for $r = 1 + \cos\theta$.

We know that the second derivative $f''(x)$ is useful in describing functions, namely, in describing concavity. We can compute $f''(x)$ in terms of polar coordinates as well. We already know how to write $dy/dx = y'$ in terms of θ, then

$$\frac{d}{dx}\frac{dy}{dx} = \frac{dy'}{dx} = \frac{dy'}{d\theta}\frac{d\theta}{dx} = \frac{dy'/d\theta}{dx/d\theta}.$$

EXAMPLE 10.2.2 We find the second derivative for the cardioid $r = 1 + \cos\theta$:

$$\frac{d}{d\theta}\frac{\cos\theta + \cos^2\theta - \sin^2\theta}{-\sin\theta - 2\sin\theta\cos\theta} \cdot \frac{1}{dx/d\theta} = \cdots = \frac{3(1+\cos\theta)}{(\sin\theta + 2\sin\theta\cos\theta)^2} \cdot \frac{1}{-(\sin\theta + 2\sin\theta\cos\theta)}$$
$$= \frac{-3(1+\cos\theta)}{(\sin\theta + 2\sin\theta\cos\theta)^3}.$$

The ellipsis here represents rather a substantial amount of algebra. We know from above that the cardioid has horizontal tangents at $\pm\pi/3$; substituting these values into the second derivative we get $y''(\pi/3) = -\sqrt{3}/2$ and $y''(-\pi/3) = \sqrt{3}/2$, indicating concave down and concave up respectively. This agrees with the graph of the function. □

Exercises 10.2.

Compute $y' = dy/dx$ and $y'' = d^2y/dx^2$.

1. $r = \theta \Rightarrow$

2. $r = 1 + \sin\theta \Rightarrow$

3. $r = \cos\theta \Rightarrow$

4. $r = \sin\theta \Rightarrow$

5. $r = \sec\theta \Rightarrow$

6. $r = \sin(2\theta) \Rightarrow$

Sketch the curves over the interval $[0, 2\pi]$ unless otherwise stated.

7. $r = \sin\theta + \cos\theta$

8. $r = 2 + 2\sin\theta$

9. $r = \dfrac{3}{2} + \sin\theta$

10. $r = 2 + \cos\theta$

11. $r = \dfrac{1}{2} + \cos\theta$

12. $r = \cos(\theta/2), 0 \le \theta \le 4\pi$

13. $r = \sin(\theta/3), 0 \le \theta \le 6\pi$

14. $r = \sin^2\theta$

15. $r = 1 + \cos^2(2\theta)$

16. $r = \sin^2(3\theta)$

17. $r = \tan\theta$

18. $r = \sec(\theta/2), 0 \le \theta \le 4\pi$

19. $r = 1 + \sec\theta$

20. $r = \dfrac{1}{1 - \cos\theta}$

21. $r = \dfrac{1}{1 + \sin\theta}$

22. $r = \cot(2\theta)$

23. $r = \pi/\theta, 0 \le \theta \le \infty$

24. $r = 1 + \pi/\theta, 0 \le \theta \le \infty$

25. $r = \sqrt{\pi/\theta}, 0 \le \theta \le \infty$