10

Polar Coordinates, Parametric Equations

10.1 Polar Coordinates

Coordinate systems are tools that let us use algebraic methods to understand geometry. While the **rectangular** (also called **Cartesian**) coordinates that we have been using are the most common, some problems are easier to analyze in alternate coordinate systems.

A coordinate system is a scheme that allows us to identify any point in the plane or in three-dimensional space by a set of numbers. In rectangular coordinates these numbers are interpreted, roughly speaking, as the lengths of the sides of a rectangle. In **polar coordinates** a point in the plane is identified by a pair of numbers (r, θ). The number θ measures the angle between the positive x-axis and a ray that goes through the point, as shown in figure 10.1.1; the number r measures the distance from the origin to the point. Figure 10.1.1 shows the point with rectangular coordinates $(1, \sqrt{3})$ and polar coordinates $(2, \pi/3)$, 2 units from the origin and $\pi/3$ radians from the positive x-axis.

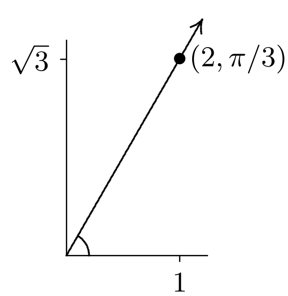

Figure 10.1.1 Polar coordinates of the point $(1, \sqrt{3})$.

Just as we describe curves in the plane using equations involving x and y, so can we describe curves using equations involving r and θ. Most common are equations of the form $r = f(\theta)$.

EXAMPLE 10.1.1 Graph the curve given by $r = 2$. All points with $r = 2$ are at distance 2 from the origin, so $r = 2$ describes the circle of radius 2 with center at the origin. □

EXAMPLE 10.1.2 Graph the curve given by $r = 1 + \cos\theta$. We first consider $y = 1 + \cos x$, as in figure 10.1.2. As θ goes through the values in $[0, 2\pi]$, the value of r tracks the value of y, forming the "cardioid" shape of figure 10.1.2. For example, when $\theta = \pi/2$, $r = 1 + \cos(\pi/2) = 1$, so we graph the point at distance 1 from the origin along the positive y-axis, which is at an angle of $\pi/2$ from the positive x-axis. When $\theta = 7\pi/4$, $r = 1 + \cos(7\pi/4) = 1 + \sqrt{2}/2 \approx 1.71$, and the corresponding point appears in the fourth quadrant. This illustrates one of the potential benefits of using polar coordinates: the equation for this curve in rectangular coordinates would be quite complicated. □

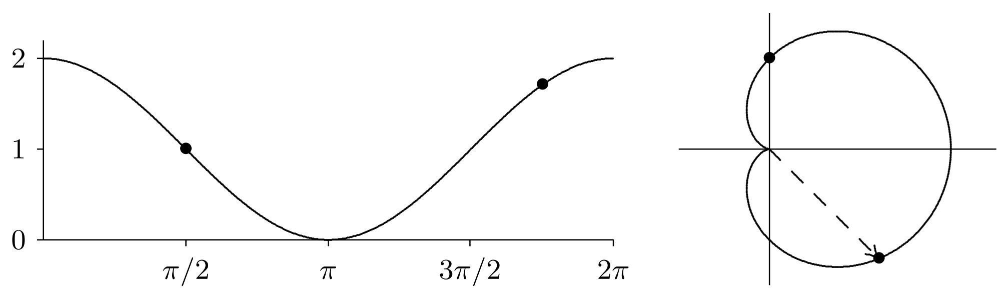

Figure 10.1.2 A cardioid: $y = 1 + \cos x$ on the left, $r = 1 + \cos\theta$ on the right.

Each point in the plane is associated with exactly one pair of numbers in the rectangular coordinate system; each point is associated with an infinite number of pairs in polar coordinates. In the cardioid example, we considered only the range $0 \le \theta \le 2\pi$, and already there was a duplicate: $(2, 0)$ and $(2, 2\pi)$ are the same point. Indeed, every value of θ outside the interval $[0, 2\pi)$ duplicates a point on the curve $r = 1 + \cos\theta$ when $0 \le \theta < 2\pi$. We can even make sense of polar coordinates like $(-2, \pi/4)$: go to the direction $\pi/4$ and then move a distance 2 in the opposite direction; see figure 10.1.3. As usual, a negative angle θ means an angle measured clockwise from the positive x-axis. The point in figure 10.1.3 also has coordinates $(2, 5\pi/4)$ and $(2, -3\pi/4)$.

The relationship between rectangular and polar coordinates is quite easy to understand. The point with polar coordinates (r, θ) has rectangular coordinates $x = r\cos\theta$ and $y = r\sin\theta$; this follows immediately from the definition of the sine and cosine functions. Using figure 10.1.3 as an example, the point shown has rectangular coordinates

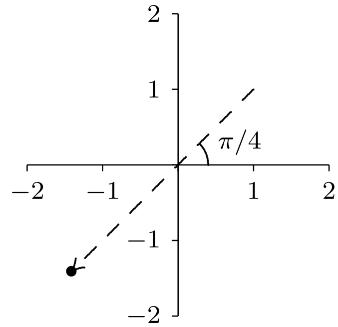

Figure 10.1.3 The point $(-2, \pi/4) = (2, 5\pi/4) = (2, -3\pi/4)$ in polar coordinates.

$x = (-2)\cos(\pi/4) = -\sqrt{2} \approx 1.4142$ and $y = (-2)\sin(\pi/4) = -\sqrt{2}$. This makes it very easy to convert equations from rectangular to polar coordinates.

EXAMPLE 10.1.3 Find the equation of the line $y = 3x + 2$ in polar coordinates. We merely substitute: $r\sin\theta = 3r\cos\theta + 2$, or $r = \dfrac{2}{\sin\theta - 3\cos\theta}$. □

EXAMPLE 10.1.4 Find the equation of the circle $(x - 1/2)^2 + y^2 = 1/4$ in polar coordinates. Again substituting: $(r\cos\theta - 1/2)^2 + r^2\sin^2\theta = 1/4$. A bit of algebra turns this into $r = \cos(t)$. You should try plotting a few (r, θ) values to convince yourself that this makes sense. □

EXAMPLE 10.1.5 Graph the polar equation $r = \theta$. Here the distance from the origin exactly matches the angle, so a bit of thought makes it clear that when $\theta \ge 0$ we get the spiral of Archimedes in figure 10.1.4. When $\theta < 0$, r is also negative, and so the full graph is the right hand picture in the figure. □

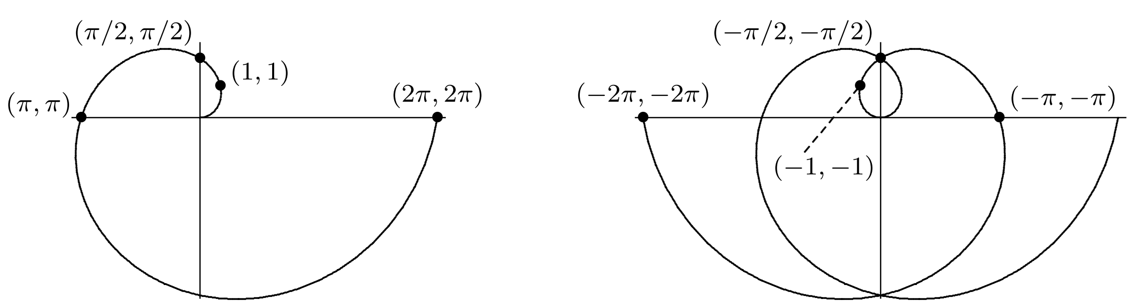

Figure 10.1.4 The spiral of Archimedes and the full graph of $r = \theta$.

Converting polar equations to rectangular equations can be somewhat trickier, and graphing polar equations directly is also not always easy.

EXAMPLE 10.1.6 Graph $r = 2\sin\theta$. Because the sine is periodic, we know that we will get the entire curve for values of θ in $[0, 2\pi)$. As θ runs from 0 to $\pi/2$, r increases from 0 to 2. Then as θ continues to π, r decreases again to 0. When θ runs from π to 2π, r is negative, and it is not hard to see that the first part of the curve is simply traced out again, so in fact we get the whole curve for values of θ in $[0, \pi)$. Thus, the curve looks something like figure 10.1.5. Now, this suggests that the curve could possibly be a circle, and if it is, it would have to be the circle $x^2 + (y-1)^2 = 1$. Having made this guess, we can easily check it. First we substitute for x and y to get $(r\cos\theta)^2 + (r\sin\theta - 1)^2 = 1$; expanding and simplifying does indeed turn this into $r = 2\sin\theta$. □

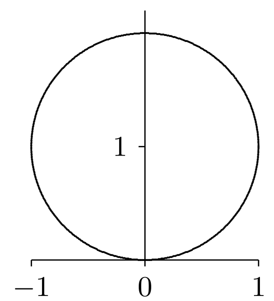

Figure 10.1.5 Graph of $r = 2\sin\theta$.

Exercises 10.1.

1. Plot these polar coordinate points on one graph: $(2, \pi/3)$, $(-3, \pi/2)$, $(-2, -\pi/4)$, $(1/2, \pi)$, $(1, 4\pi/3)$, $(0, 3\pi/2)$.

Find an equation in polar coordinates that has the same graph as the given equation in rectangular coordinates.

2. $y = 3x$ ⇒
3. $y = -4$ ⇒
4. $xy^2 = 1$ ⇒
5. $x^2 + y^2 = 5$ ⇒
6. $y = x^3$ ⇒
7. $y = \sin x$ ⇒
8. $y = 5x + 2$ ⇒
9. $x = 2$ ⇒
10. $y = x^2 + 1$ ⇒
11. $y = 3x^2 - 2x$ ⇒
12. $y = x^2 + y^2$ ⇒

Sketch the curve.

13. $r = \cos\theta$
14. $r = \sin(\theta + \pi/4)$
15. $r = -\sec\theta$
16. $r = \theta/2$, $\theta \ge 0$
17. $r = 1 + \theta^1/\pi^2$
18. $r = \cot\theta\csc\theta$
19. $r = \dfrac{1}{\sin\theta + \cos\theta}$
20. $r^2 = -2\sec\theta\csc\theta$

Find an equation in rectangular coordinates that has the same graph as the given equation in polar coordinates.

21. $r = \sin(3\theta)$ ⇒

22. $r = \sin^2\theta$ ⇒

23. $r = \sec\theta\csc\theta$ ⇒

24. $r = \tan\theta$ ⇒