Application: Work

8.1 What is Work?

Work, in the physics sense, is usually defined as 'force acting over a distance.' Work is *sometimes* force times distance,[1] but not always. Work is more subtle than that. Every time you exert a force, it is not the case that any work is done (even though it may feel like that to you!

[1] Work will be force times distance in all of the applications we consider.

Why? Well, the work/energy equation says that work done (by the net force on an object) equals the object's change in kinetic energy. More simply:

$$\text{Work} = \text{Change in Kinetic Energy.} \tag{8.1}$$

This means that if an object's kinetic energy doesn't change, then no work has been done on the object—whether or not a force has been exerted. In particular, a force will do work only if the force has a component in the direction that the object moves.

Here's the distinction: if you push on a big box and it moves in the direction you push, then work is accomplished and it equals force times distance. If the box is very big and when you push it nothing happens (it does not move), then no work is done. If a force is applied to the box in a certain direction and the box moves, but not in the direction you push (maybe your friend is pushing on another side) but in some other direction, then work is done, but the amount depends on the angle the box moves relative to the direction the box moves. So again the answer is not simply force times distance.

However, to keep things simple, in the context of this section we will assume the direction the force is applied and the direction of the motion of the object are one and the same. In this case, *assuming that the force is constant*,

$$\boxed{W = \text{Work} = \text{force} \times \text{Distance} = F \cdot x.}$$

The unit used to measure work vary depending on the system you are in—and they are probably less familiar to you than the units for velocity and acceleration. Table 8.1 lists the units for three common systems.

System	**Force**	**Distance**	**Work**
British	pounds	feet	foot-pounds (ft-lbs)
cgs	dynes	centimeters	ergs
SI (international)	Newtons	meters	joules

Table 8.1: Units of work.

EXAMPLE 8.1. Calculate the work done in lifting a 147 lb object 22 feet (e.g., me walking from the first to the third floor in Lansing)? Well,

$$\text{Work} = \text{force} \times \text{Distance} = 147 \times 22 = 3,234 \text{ ft-lbs.}$$

That was easy.

Now the key thing to notice is that under our assumptions, work is a *product*. One of the assumptions is that a constant force is applied. But most forces are not constant. Consider the following situation.

EXAMPLE 8.2. Suppose we want to hoist a leaking bucket vertically. Because of the leak, the force applied to lift the bucket decreases, so that

$$F(x) = 60\left(1 - \frac{x^2}{5000}\right), \qquad 0 \le x \le 50$$

where x is measured in feet. Find the work done in lifting the bucket.

So how do we calculate the work done if the force varies? Well, remember that Riemann sums involve products. So we will use the 'subdivide and conquer' strategy once more. Unlike in the earlier cases there is no natural figure to draw whose area, volume, or arc length we are trying to calculate. This time we must apply the theory.

General Situation. Assume that $F(x)$ is a variable but continuous force that is a function of the position x and that it is applied over an interval $[a, b]$. Find the work done. As usual, let $P = \{x_0, x_1, \ldots, x_n\}$ be a regular partition of $[a, b]$ into n equal width subintervals of length Δx. Now if the intervals are short enough, since the force is continuous, the force will be nearly constant on each interval, though its value will vary from interval to interval. So let W_i denote the work done on the ith subinterval. Since the length of the ith subinterval is Δx, then

$$W_i \approx F(x_i)\Delta x.$$

Consequently, adding up the 'pieces of work' on each subinterval,

$$\text{Total Work} = \sum_i^n W_i \approx \sum_i^n F(x_i)\Delta x. \tag{8.2}$$

Notice that we now have a Riemann sum involving the force function! To improve the approximation we do the standard thing: We let the number of subdivisions get large and take the limit. We find

$$\text{Total Work} = \lim_{n\to\infty} \sum_i^n W_i = \lim_{n\to\infty} \sum_i^n F(x_i)\Delta x = \int_a^b F(x)\,dx.$$

We are certain that this limit of the Riemann sums exists and is, in fact, the definite integral because we assumed that the force $F(x)$ is continuous function of the position.

THEOREM 8.1 (Work Formula). If $F(x)$ is a continuous force that is a function of the position x that is applied over an interval $[a, b]$, then the **work** done over the interval is

$$\text{Work} = \int_a^b F(x)\,dx.$$

Stop and Step Back. There are a couple of things I want you to notice. First, Theorem 8.1 amounts to saying that work is the 'area under the force curve.' That's probably not how you would first think of it, but that's what the theorem says! Second, when I was writing these notes, I simply cut-and-pasted the earlier material on arc length and changed the a few words here and there. But it is the same 'subdivide and conquer' strategy that you have seen several times now. You should be able to create such arguments for yourself now.

EXAMPLE 8.3 (Return to the leaky bucket)**.** Return to the leaking bucket. The force applied was continuous and it was applied on the

$$F(x) = 60\left(1 - \frac{x^2}{5000}\right),$$

and it was applied on the interval $[0, 50]$ where x is measured in feet. So by Theorem 8.1 the work done in lifting the bucket is

$$\begin{aligned}
\text{Work} = \int_a^b F(x)\,dx &= \int_0^{50} \left(1 - \frac{x^2}{5000}\right)\,dx \\
&= 60\left(x - \frac{x^3}{15,000}\right)\Bigg|_0^{50} \\
&= 60\left(50 - \frac{125,000}{15,000}\right) - 0 \\
&= 2500 \text{ ft-lbs.}
\end{aligned}$$

Pretty straightforward!

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